HSC Mathematics (2 Unit)

SAMPLE LECTURE SLIDES

HSC Exam Preparation Programs
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1. Sequences & Series Applications

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1.1 Arithmetic Sequences

A *sequence* is an ordered set of numbers. Each number is called a *term* of the sequence. Sequences can be finite or infinite. Many sequences contain terms that vary according to some rule or pattern.

In an *arithmetic* sequence, the difference, d, between successive terms is constant. For example, $3, 5, \ldots, 19$. If we call the first term a, then we have

$$\{a,\ a+d,\ a+2d,\ \ldots,\ a+(n-1)d\}$$

The nth term, T_n, is given by

$$\boxed{T_n = a + (n-1)d}$$

For example, find the sixth term for the arithmetic sequence $\{4, 7, 10, \ldots\}$.

$$a = 4, \quad d = 7 - 4 = 3 \qquad\qquad \implies T_6 = 4 + (6-1) \times 3 = 19$$

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As another example, find the first positive term of $\{-50, -47, -44, \ldots\}$.

First note that $a = -50$ and $d = 3$. For the first positive term, $T_n > 0$

$$\begin{aligned} \text{i.e. } -50 + (n-1)3 &> 0 \\ -50 + 3n - 3 &> 0 \\ 3n &> 53 \\ n &> 17.67 \end{aligned}$$

$\therefore n = 18$ gives the first positive term.

Hence, the first positive term is $T_{18} = -50 + 17 \times 3 = 1$.

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1.1 Arithmetic Sequences

As final example, find the first term and common difference of the arithmetic sequence whose fifth and twelfth terms are 31 and 73, respectively.

Both statements in the question form separate equations that can be solved simultaneously for a and d.

$$T_5 = a + 4d = 31$$
$$T_{12} = a + 11d = 73$$

By subtracting, we get

$$7d = 42$$
$$\therefore d = 6$$

Substituting $d = 6$ into the first equation, we have

$$a + 4 \times 6 = 31$$
$$\therefore a = 7$$

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1.1 Arithmetic Sequences

The sum of the n terms of a sequence, S_n, is called a *series*. An arithmetic series is given by the formula

$$\boxed{S_n = \frac{n}{2}\left(2a + (n-1)d\right)}$$

To prove this we can write the sum of n terms twice, once forwards and once backwards, then add the two equations together to solve for S_n.

$$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) \quad \textbf{(1)}$$

$$S_n = (a+(n-1)d) + (a+(n-2)d) + (a+(n-3)d) + \cdots + a \quad \textbf{(2)}$$

$$2S_n = (2a+(n-1)d) + (2a+(n-1)d) + \cdots + (2a+(n-1)d) \quad \textbf{(1)+(2)}$$

$$2S_n = n(2a+(n-1)d)$$

$$\therefore S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$

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For example, find the sum of 15 terms of the series $2 + 8 + 14 + \ldots$.

Here $a = 2$ and $d = 6$. Using our formula, we have

$$\begin{aligned} S_n &= \frac{n}{2}\left(2a + (n-1)d\right) \\ S_{12} &= \frac{12}{2}(2 \times 2 + 11 \times 6) \\ &= 6(4 + 66) \\ &= 420 \end{aligned}$$

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1.1 Arithmetic Sequences

As another example, find the value of $9 + 14 + 19 + \cdots + 224$.

Here $a = 9$ and $d = 5$. We should find what term 224 corresponds to.

$$\begin{aligned} T_n &= 9 + (n-1)5 \\ 224 &= 9 + 5n - 5 \\ 220 &= 5n \\ n &= 44 \end{aligned}$$

Now we know we are evaluating S_{44}, which is given by

$$\begin{aligned} S_{224} &= \frac{224}{2}(2 \times 9 + 223 \times 5) \\ &= 5126 \end{aligned}$$

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1.1 Arithmetic Sequences

As a final example, find how many terms in the series $48 + 44 + 40 + \ldots$ need to be included such that their sum is 308.

Here $a = 48$ and $d = -4$. Our aim is to find what n corresponds to $S_n = 308$.

$$S_n = \frac{n}{2}(2 \times 48 + (n-1)(-4))$$
$$308 = \frac{n}{2}(96 + 4n + 4)$$
$$616 = n(100 + 4n)$$
$$616 = 4n(25 + n)$$
$$154 = 25n + n^2$$

We now have a quadratic equation, which can be solved by factorising.

$$n^2 + 25n - 154 = 0$$
$$(n - 11)(n - 14) = 0$$
$$n = 11 \text{ or } 14$$

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1.2 Geometric Sequences

In a geometric sequence, the ratio, r, between successive terms is constant. If we call the first term a, then we have

$$\{a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1}\}$$

The nth term, T_n, is given by

$$\boxed{T_n = ar^{n-1}}$$

The sum of n terms, S_n, is found using

$$\boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$$

For example, find T_{10} and S_{10} for the geometric sequence $\{2, 6, 18, \ldots\}$.

$$a = 2 \quad \Longrightarrow \quad T_6 = 2 \times 3^5 \quad \text{and} \quad S_{10} = \frac{2 \times (3^{10} - 1)}{3 - 1}$$

$$r = \frac{6}{2} = 3 \qquad\qquad = 486 \qquad\qquad = 59,048$$

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As another example, find which term of the series $4 + 12 + 36 + \ldots$ is equal to $78,732$.

Here $a = 4$ and $r = 3$. We want to find the n corresponding to $T_n = 78,732$.

$$\begin{aligned} T_n &= ar^{n-1} \\ 78,732 &= 4 \times 3^{n-1} \\ 19,683 &= 3^{n-1} \\ \log_{10} 19,683 &= (n-1)\log_{10} 3 \\ \frac{\log_{10} 19,683}{\log_{10} 3} &= n - 1 \\ 9 &= n - 1 \\ n &= 10 \end{aligned}$$

Hence, it is the 10th term that is $78,732$.

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1.2 Geometric Sequences

As a final example, evaluate $\sum_{n=1}^{8} 3 \times 2^{n-1}$.

This is an example of sigma (Σ) notation. $\sum_{n=1}^{8}$ means evaluate the expression with $n = 1$, again with $n = 2$ etc. until $n = 8$, then add all the terms together'.

Hence, $\sum_{n=1}^{8} 3 \times 2^{n-1}$ is the series $3 + 6 + 12 + \cdots + 384$.

Since this is a geometric series, we can use

$$S_n = \frac{a(r^n - 1)}{r - 1}, \text{ where } a = 3, \; r = 2, \; n = 8$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{8} 3 \times 2^{n-1} = S_8 &= \frac{3(2^8 - 1)}{2 - 1} \\ &= 765 \end{aligned}$$

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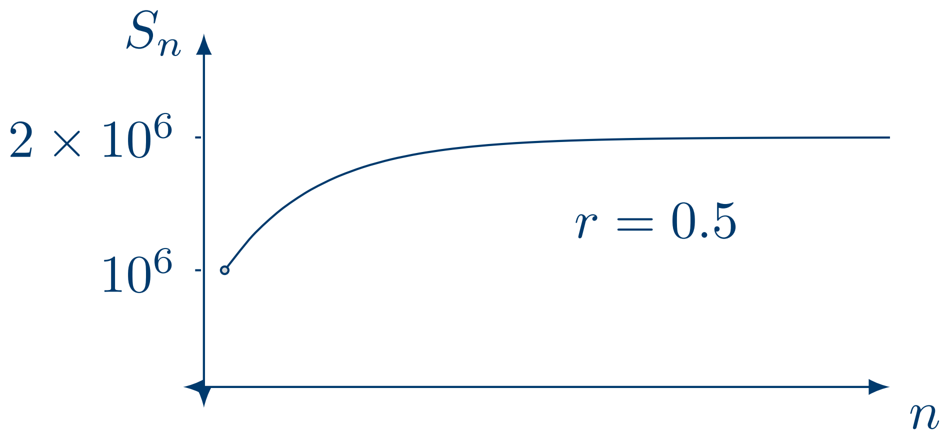
1.3 Infinite Series

If $-1 < r < 1$, then a *limiting sum* exists for the geometric series, *i.e.* the series converges to a finite number as $n \to \infty$. The limiting sum, S_∞, is found using

$$S_\infty = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n$$

$$\boxed{S_\infty = \frac{a}{1-r}}$$



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1.4 Compound Interest

The future value, A, of an initial principal of money, P, when invested at $r\%$ per period for a duration of n periods is found using

$$\boxed{A = P\left(1 + \frac{r}{100}\right)^n}$$

For example, \$7,000 is invested at 4.8% p.a. compounded quarterly. How long until the balance is greater than \$14,000?

- Data: $P = 7,000$, $r = \frac{4.8}{4}\% = 1.2\%$ (per quarter), $A > 14,000$.

$$\begin{aligned} 7,000\left(1 + \tfrac{1.2}{100}\right)^n &> 14,000 \\ 1.012^n &> 2 \\ n\ln(1.012) &> \ln 2 \\ n &> \tfrac{\ln(2)}{\ln(1.012)} \\ &> 58.1 \text{ quarters, or } 4.8 \text{ years} \end{aligned}$$

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1.5 Superannuation

Superannuation problems combine geometric series with compound interest.

For example, \$900 is invested at the beginning of each year into an account earning 4% p.a. compounded annually. How much is this investment worth at the end of 5 years?

- The first \$900 is invested for 5 years, *i.e.* $A_1 = 900 \times 1.04^5$.
- The second \$900 is invested for 4 years, *i.e.* $A_2 = 900 \times 1.04^4$.
- The fifth \$900 is invested for 1 year, *i.e.* $A_5 = 900 \times 1.04$.

$$\begin{aligned} A_1 + A_2 + A_3 + A_4 + A_5 &= 900 \times 1.04^5 + 900 \times 1.04^4 + \cdots + 900 \times 1.04 \\ &= 900 \times 1.04 \left(\overbrace{1}^{a} + \overbrace{1.04}^{r} + \cdots + 1.04^4 \right) \\ &= 936 \left(\frac{1.04^5 - 1}{1.04 - 1} \right) = \$5,069.68 \end{aligned}$$

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1.6 Loan Repayments

Loan problems are similar to superannuation, except that regular payments, M, are made to reduce the initial owed amount, P. The amount owing after n repayments is denoted A_n.

- Step 1: Construct a formula for P, M and A_n.
- Step 2: Simply the geometric series using $S_n = \frac{a(r^n-1)}{r-1}$.
- Step 3: To find M, equate A_n to zero and solve.

For a loan of $\$P$ taken out over n time periods at a reducible interest rate of $r\%$ per period, the formula for the regular repayment, $\$M$, is given by

$$M = P\left(1 + \frac{r}{100}\right)^n \left[\frac{\frac{r}{100}}{(1+\frac{r}{100})^n - 1}\right]$$

The syllabus requires you to be able to derive this, rather than memorise it. Fortunately, constructing the formula almost always follows the same 3 steps.

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For example, \$2,000 is borrowed at the reducible interest rate of 18% p.a. (1.5% per month). What should be the monthly repayment, $\$M$, to pay the loan off in 2 years?

- At the end of the 1st month, the \$2,000 will have grown by 1.5% and had a repayment subtracted from it, *i.e.* $A_1 = 2,000 \times 1.015 - M$.
- After 2 months,

$$\begin{aligned} A_2 &= (2,000 \times 1.015 - M) \times 1.015 - M \\ &= 2,000 \times 1.015^2 - M(1 + 1.015) \end{aligned}$$

- After 24 months,

$$\begin{aligned} A_{24} &= 2,000 \times 1.015^{24} - M(\overbrace{1 + 1.015 + \ldots 1.015^{23}}^{S_{24} \text{ where } a=1 \text{ \& } r=1.015}) \\ &= 2,000 \times 1.015^{24} - M\left(\frac{1 \times (1.015^{24} - 1)}{1.015 - 1}\right) \end{aligned}$$

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- Equate A_{24} to zero, *i.e.* we aim to pay the loan off after 24 months,

$$0 = 2,000 \times 1.015^{24} - M\frac{(1.015^{24}-1)}{0.015}$$

$$M\frac{(1.015^{24}-1)}{0.015} = 2,000 \times 1.015^{24}$$

$$M = \frac{0.015 \times 2,000 \times 1.015^{24}}{1.015^{24}-1}$$

$$M = 99.85$$

Hence, to pay off the \$2,000 loan after 2 years, our monthly repayments, \$$M$, are \$99.85.

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Question 1 (6 Marks)

Alice and Bob work at two different companies. At the end of this year, Alice will have earned $40,000. Each year she stays on is accompanied by a 3% pay-rise. Bob's salary is $43,000 but enjoys a $2,200 pay-rise every other year, starting in the third year.

(i) Who earns a higher salary at the end of the 8th year? (3 Marks)

(ii) What is the difference in their total earnings after 8 years? (3 Marks)

Solution

(i) Alice's salary follows a geometry sequence, with $a = \$40,000$ and $r = 1.03$. Her salary in the 8th year is

$$\begin{aligned} T_8 &= \$40,000(1.03)^8 \\ &= \$50,671 \end{aligned}$$

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1.7 HSC-Adapted Questions

Bob's salary follows an arithmetic sequence, with $a = \$43,000$ and $d = \$2,200$, but the time period is every 2 years. Therefore, his salary in the 8th year is the 4th term in the arithmetic sequence, which is given by

$$\begin{aligned} T_4 &= \$43,000 + (4-1) \times \$2,200 \\ &= \$49,600 \end{aligned}$$

Hence, in the eighth year Alice earns \$1,071 more than Bob.

(ii) The sum of Alice's first 8 salaries is,

$$\begin{aligned} S_8 &= \frac{\$40,000(1.03^8 - 1)}{1.03 - 1} \\ &= \frac{\$40,600}{0.03}(1.03^8 - 1) \\ &= \$361,029 \end{aligned}$$

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For Bob, the year following a pay-rise is a repeated salary. Hence, after 8 years, Bob has 4 unique salaries defined by the arithmetic sequence with $a = \$43,000$ and $d = \$2,200$. The sum of Bob's first 8 salaries is therefore,

$$
\begin{aligned}
2 \times S_4 &= 2 \times \frac{3}{2}(2 \times \$43,000 + 3 \times \$2,200) \\
&= 3 \times \$92,600 \\
&= \$277,800
\end{aligned}
$$

Hence, after eight years Alice earns $83,229 more than Bob.

Question 2 (6 Marks)

A $400,000 loan is to be repaid in equal monthly repayments, $\$M$, over 35 years. The reducible interest rate of 7.5% is calculated monthly.

(i) Calculate the monthly repayment, $\$M$. (3 Marks)

(ii) After how long will the amount owing be less than $100,000? (3 Marks)

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Solution

(i) Let $\$A_n$ be the amount owing after the nth repayment. The monthly interest rate is $\frac{7.5}{12}\% = 0.625\%$. After 1 month,

$$A_1 = 400,000 \times 1.00625 - M$$

After 2 months,

$$\begin{aligned} A_2 &= (400,00 \times 1.00625 - M) \times 1.00625 - M \\ &= 400,000 \times 1.00625^2 - M \times 1.00625 - M \\ &= 400,000 \times 1.00625^2 - M(1 + 1.00625) \end{aligned}$$

After $35 \times 12 = 420$ months,

$$\begin{aligned} A_{420} &= 400,000 \times 1.00625^{420} - M(1 + 1.00625 + 1.00625 + \cdots + 1.00625^{419}) \\ &= 400,000 \times 1.00625^{420} - M\left(\frac{1.00625^{420} - 1}{1.00625 - 1}\right) \end{aligned}$$

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To find $\$M$, substitute $\$A_{420} = \$0$, *i.e.* no amount owing after 420 months.

$$0 = 400,000 \times 1.00625^{420} - M\left(\frac{1.00625^{420} - 1}{0.00625}\right)$$

$$M = \frac{0.00625 \times 400,000 \times 1.00625^{420}}{1.00625^{420} - 1}$$

$$= 2,696.97$$

(ii) Now that $\$M$ is known, it can be substituted into an expression for $\$A_n$.

$$400,000 \times 1.00625^n - 2,696.97\left(\frac{1.00625^n - 1}{0.00625}\right) < 100,000$$

$$400,000 \times 1.00625^n - 431,515.20(1.00625^n - 1) < 100,000$$

$$400,000 \times 1.00625^n - 431,515.20 \times 1.00625^n + 431,515.20 < 100,000$$

$$1.00625^n(400,000 - 431,515.20) < -331,515.20$$

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When we rearrange the expression, the inequality needs to be flipped because we are dividing by a negative quantity.

$$\begin{aligned} 1.00625^n &> \frac{-331,515.20}{-31,515.20} \\ &> 10.5192 \end{aligned}$$

Taking logs of both sides, we have

$$\begin{aligned} n\ln(1.00625) &> \ln(10.5192) \\ n &> \frac{\ln(10.5192)}{\ln(1.00625)} \\ &> 377.7 \end{aligned}$$

Hence, the balance will be less than $100,000 after 378 months (31.5 years).

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