HSC Mathematics Ext. 1 (3 Unit)

SAMPLE LECTURE SLIDES

HSC Exam Preparation Programs
26 September 2015

Overview

1. Further Trigonometry
2. Circle Geometry
3. Parametric Equations
4. Mathematical Induction
5. Polynomials
6. Binomial Theorem
7. Further Probability
8. Integration Methods
9. Inverse Trigonometric Functions
10. Rates of Change
11. Rectilinear Motion
12. Projectile Motion
13. Simple Harmonic Motion

10. Rates of Change

10.1 Chain Rule Applications

In the 2 Unit course, we learned that 'the rate of change' of a function, $Q(t)$, means 'the derivative with respect to time', i.e. $\frac{\mathrm{d}Q}{\mathrm{d}t}$.

In the 3 Unit course, Q may not be explicitly known as a function of t but of another variable, u, instead. Hence, $\frac{\mathrm{d}Q}{\mathrm{d}t}$ must be found using the Chain Rule.

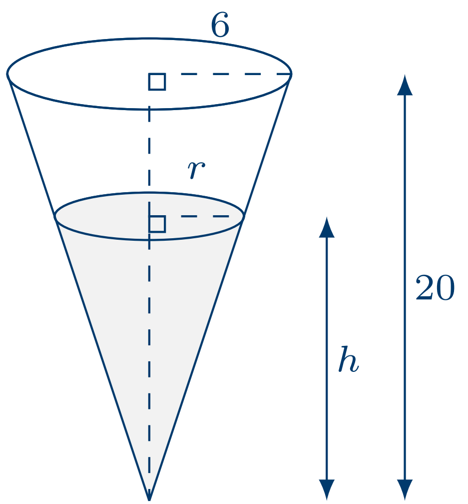
$$\boxed{\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}Q(u)}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}t}}$$

- Step 1: Write down the known $\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)$ & unknown $\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)$ time-derivatives.
- Step 2: Find the connection between Q and u. We need it explicitly as $Q(u)$, with no other variables in the expression.
- Step 3: Differentiate $Q(u)$ to get $\frac{\mathrm{d}Q}{\mathrm{d}u}$.
- Step 4: Substitute your values for $\frac{\mathrm{d}Q}{\mathrm{d}u}$ and $\frac{\mathrm{d}u}{\mathrm{d}t}$ into the Chain Rule to solve for $\frac{\mathrm{d}Q}{\mathrm{d}t}$.

10.1 Chain Rule Applications

For example, an inverted cone of base radius 6 cm and height 20 cm has water flowing from its apex at the constant rate of 6 cm^3/s.

Use the Chain Rule to determine the rate at which the water level is falling when the water level is 4 cm.



- Step 1: We want $\frac{dh}{dt}$ when h is 4 cm. We know $\frac{dV}{dt}$ is -6 cm^3/s.

- Step 2: For a cone, V and h are related by $V = \frac{1}{3}\pi r^2 h$.

  Since r is also a variable, we need to rearrange the expression so it contains only V and h as variables. Notice the geometric relationship between r and h in the diagram: equiangular similar triangles means the ratio $\frac{r}{6}$ is equal to $\frac{h}{20}$.

  Hence,

$$r = 6 \times \frac{h}{20} \quad \Longrightarrow \quad r = \frac{3h}{10}$$

  Substituting this into our volume expression gives,

$$V = \frac{1}{3}\pi \left(\frac{3h}{10}\right)^2 h$$
$$\therefore \; V = \frac{3\pi}{100}h^3$$

10.1 Chain Rule Applications

- Step 3: Differentiating $V = \frac{3\pi}{100}h^3$ gives us,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{9\pi}{100}h^2 \quad \text{or} \quad \frac{\mathrm{d}h}{\mathrm{d}V} = \frac{100}{9\pi h^2}$$

- Step 4: Substituting our values for $\frac{\mathrm{d}h}{\mathrm{d}V}$ and $\frac{\mathrm{d}V}{\mathrm{d}t}$ into the Chain Rule,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}$$
$$\therefore \frac{\mathrm{d}h}{\mathrm{d}t} = \left(\frac{100}{9\pi h^2}\right) \cdot \left(-6 \text{ cm}^3/\text{s}\right)$$

Lastly, at a water level of 4 cm, our expression simplifies to

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{200 \text{ cm}^3/\text{s}}{3\pi\left(4 \text{ cm}\right)^2} \quad \text{when } h = 4 \text{ cm}$$
$$\therefore \frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{25}{6\pi} \text{ cm/s}$$

10.2 Newton's Law of Cooling

This is an extension on *Exponential Growth and Decay* discussed in the 2 Unit course.

Newton showed that an object's temperature is governed by,

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_f)}$$

where $k > 0$ is a constant and T_f is the equilibration (final) temperature.

The solution to this equation is found by integrating the inverse expression.

That is, we first invert Newton's differential equation,

$$\frac{\mathrm{d}t}{\mathrm{d}T} = \frac{-1}{k(T - T_f)}.$$

10.2 Newton's Law of Cooling

Secondly, we integrate the fraction to form a log,

$$t = \int \frac{-1}{k(T - T_f)}\,\mathrm{d}T = \frac{-1}{k}\ln(T - T_f) + C.$$

Subtracting C and multiplying by $-k$, give us,

$$-kt + kC = \ln(T - T_f).$$

We can now exponentiate both sides to arrive at

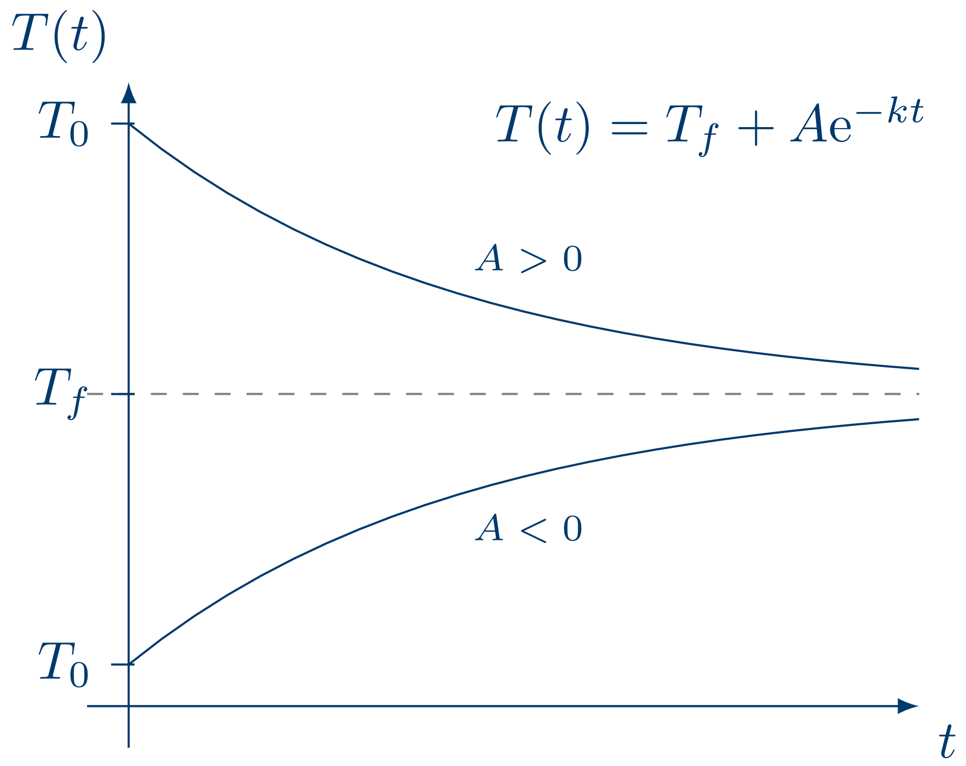
$$\mathrm{e}^{-kt+kC} = T - T_f \quad \text{or } A\mathrm{e}^{-kt} = T - T_f,$$

if we define a simplified constant as $A = \mathrm{e}^{kC}$. Solving for T gives us,

$$\boxed{T(t) = T_f + A\mathrm{e}^{-kt}}$$

10.2 Newton's Law of Cooling

The constant A represents the difference between initial and final temperatures. To see this, substitute $t = 0$ into $T(t)$ to arrive at $T(0) = T_f + A$, or $A = T_0 - T_f$.



10.2 Newton's Law of Cooling

For example, given a cooling constant of 0.06 min^{-1}, how much quicker is it to chill 18°C tap water to 7°C when using a 5°C refrigerator compared with a -20°C freezer?

- Refrigerator data: $T(t) = 7°\text{C}$, $T_0 = 18°\text{C}$, $T_f = 5°\text{C} \therefore\ A = 13°\text{C}$.
- Step 1: Substitute the data into Newton's temperature formula.

$$7 = 5 + 13e^{-0.06t}$$

$$\therefore\ \frac{2}{13} = e^{-0.06t}$$

- Step 2: Solve for t by taking logs of both sides,

$$\ln\left(\frac{2}{13}\right) = -0.06t$$

$$\therefore\ t = -\frac{1}{0.06}\ln\left(\frac{2}{13}\right) = 31.2 \text{ min}$$

10.2 Newton's Law of Cooling

- Freezer data: $T(t) = 7°$C, $T_0 = 18°$C, $T_f = -20°$C $\therefore\ A = 38°$C.
- Step 1: Substitute the data into Newton's temperature formula.

$$7 = -20 + 38e^{-0.06t}$$
$$\therefore\ \frac{27}{38} = e^{-0.06t}$$

- Step 2: Solve for t by taking logs of both sides,

$$\ln\left(\frac{27}{38}\right) = -0.06t$$
$$\therefore\ t = -\frac{1}{0.06}\ln\left(\frac{27}{38}\right) = 5.7 \text{ min}$$

Hence, the freezer is 26 minutes faster than the refrigerator at chilling the water from 18°C to 7°C.

Quesiton 1 (4 Marks)

Alice, Bob, and Charlie stand so as to form a right-angled triangle, with Charlie and Bob forming the hypotenuse and Alice and Bob separated by 20 metres. Charlie proceeds to walk towards Alice at a constant rate of change of 2 metres per second. At what rate is the distance between Charlie and Bob changing, when Charlie is 15 metres from Alice?

Solution

Let the distances between Charlie and Alice and Charlie and Bob be a and b, respectively. We need to calculate $\frac{\mathrm{d}b}{\mathrm{d}t}$ when a is 15 metres. We know that $\frac{\mathrm{d}a}{\mathrm{d}t}$ is -2 metres per second, since the distance is decreasing.

Using Pythagorus' Theorem,

$$20^2 + a^2 = b^2$$

$$\therefore \quad b = \sqrt{400 + a^2}$$

Differentiating this expression, we have

$$\begin{aligned}\frac{\mathrm{d}b}{\mathrm{d}a} &= \frac{\mathrm{d}}{\mathrm{d}a}\left(400+a^2\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\times\left(400+a^2\right)^{-\frac{1}{2}}\times 2a \\ &= \frac{a}{\sqrt{400+a^2}} \\ &= \frac{3}{5}, \quad \text{when } a = 15\end{aligned}$$

Substituting these values into the Chain Rule, we arrive at

$$\begin{aligned}\frac{\mathrm{d}b}{\mathrm{d}t} &= \frac{3}{5}\times -2 \\ &= -\frac{6}{5}\end{aligned}$$

Hence, when Charlie is 15 m from Alice, he approaches Bob at 1.2 m/s.

Quesiton 2 (3 Marks)

A cup of tea has an initial temperature of 100°C. The temperature, T°C, of the tea after t minutes is given by

$$T(t) = X + Y\mathrm{e}^{-kt},$$

where X, Y, and k are positive constants.

In a room with an ambient temperature of 19°C, the temperature of the tea drops to 85°C after 5 minutes. How long does the tea take to cool to 50°C?

Solution

The final temperature of the tea will be 19°C, $\therefore \ X = 19°\text{C}$.

Also, we are told that $T(0) = 100°\text{C}$, $\therefore \ Y = (100 - 19)°\text{C} = 81°\text{C}$.

10.3 HSC-Adapted Questions

When $t = 5$, $T = 85°\text{C}$. Substituting this into the equation, we have

$$80 = 19 + 81\mathrm{e}^{-k\times 5}$$

$$61 = 81\mathrm{e}^{-5k}$$

$$\frac{61}{81} = \mathrm{e}^{-5k}$$

$$\therefore \quad k = -\frac{1}{5}\ln\left(\frac{61}{81}\right)$$

Now that all the constants are known, we can substitute $T = 50°\text{C}$.

$$50 = 19 + 81\mathrm{e}^{\frac{1}{5}\ln\left(\frac{61}{81}\right)t}$$

$$\frac{31}{81} = \mathrm{e}^{\frac{1}{5}\ln\left(\frac{61}{81}\right)t}$$

$$\therefore \quad t = \frac{\ln\left(\frac{31}{81}\right)}{\frac{1}{5}\ln\left(\frac{61}{81}\right)} = 16.9 \text{ minutes}$$

Quesiton 3 (6 Marks)

At 3pm in a school playground, a melted chocolate bar is found to be at a temperature of x°C. The packaging on the chocolate bar says it will begin to melt at 16°C. The temperature, T°C, of the bar t minutes after 3pm varies according to the differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{60}\ln(1.6)(A - T),$$

where A is a constant.

(i) Show that for any constant, B, a solution to the differential equation is

$$T = A + Be^{-\frac{1}{60}\ln(1.6)t}.$$

(ii) After an hour, the chocolate bar increases in temperature by $\frac{15}{4}$°C. Given the chocolate bar started to melt at 2pm, find x and the limiting temperature of the bar. Assume the temperature of the day is constant.

Solution

(i) Differentiating the proposed solution, we have

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(A + B\mathrm{e}^{-\frac{1}{60}\ln(1.6)t}\right)\\ &= B\mathrm{e}^{-\frac{1}{60}\ln(1.6)t} \times -\frac{1}{60}\ln(1.6)\\ &= (T - A) \times -\frac{1}{60}\ln(1.6)\\ &= \frac{1}{60}\ln(1.6)(A - T)\end{aligned}$$

as required.

(ii) We are told that $T(0) = x°$C, $\therefore\ x = A + B$.

When $t = -60$, $T = 16°$C. Substituting these values into the equation,

$$16 = A + Be^{-\frac{1}{60}\ln(1.6)\times -60}$$
$$16 = A + Be^{\ln(1.6)}$$
$$16 = A + B \times 1.6$$
$$\therefore \quad B = \frac{16 - A}{1.6}$$

When $t = 60$, $T = (x + \frac{15}{4})°$C. Hence, we have

$$x + \frac{15}{4} = A + Be^{-\frac{1}{60}\ln(1.6)\times 60}$$
$$A + B + \frac{15}{4} = A + Be^{-\ln\left(\frac{16}{10}\right)}$$
$$B + \frac{15}{4} = B \times \frac{10}{16}$$
$$\therefore \quad B = -10$$

Combining these equations, we have

$$\frac{16 - A}{1.6} = -10$$
$$\therefore \quad A = 16 + 16 = 32$$

Since $x = A + B$, we have

$$x = 32 - 10$$
$$\therefore \quad x = 22$$

The limiting temperature occurs when $t \to \infty$. Since $\mathrm{e}^{-\frac{1}{60}\ln(1.6)t} \to 0$ as $t \to \infty$, we arrive at

$$T(t \to \infty) = 32 - 10 \times 0$$
$$= 32$$

which is the equilibrium temperature.