HSC Mathematics Ext. 2 (4 Unit)

SAMPLE LECTURE SLIDES

HSC Exam Preparation Programs
27 September 2015

Overview

1. Graphs & Curve Sketching
2. Conics
3. Polynomials
4. Complex Numbers
5. Complex Locus Problems
6. Integration
7. Volumes
8. Mechanics

6. Integration

6.1 Integration by Substitution

This technique is derived from the *Fundamental Theorem of Calculus*, which states that, for continuous functions $f(x) = F'(x)$ and $u(x)$,

$$\int_a^b (F(u))' \, \mathrm{d}x = F\left(u(b)\right) - F\left(u(a)\right)$$
$$= \int_{u(a)}^{u(b)} f(u) \, \mathrm{d}u.$$

The *Chain Rule* tell us that

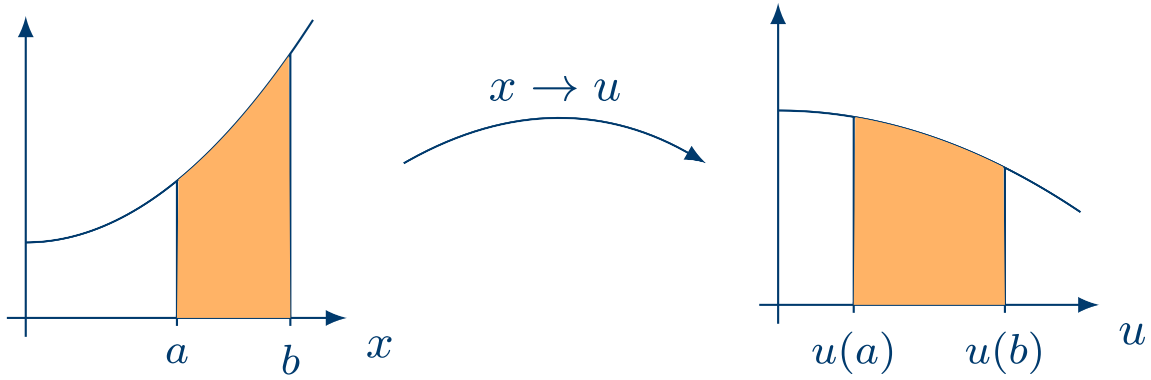
$$(F(u))' = f(u)u'.$$

Hence, we arrive at the rule for Integration by Substitution:

$$\boxed{\int_a^b g(x) \, \mathrm{d}x = \int_{u(a)}^{u(b)} f(u) \, \mathrm{d}u, \text{where } g(x) = f(u)u'}$$

6.1 Integration by Substitution

The original integral (in x) is transformed to an equivalent integral in u.



For example, solve $\int_{-1}^{1} x\sqrt{2x+3}\,\mathrm{d}x$ using the substitution $u = 2x + 3$.

- Step 1: Find an expression for u'.

$$u = 2x + 3 \implies u' = 2$$

- Step 2: Substitute every x with u in the integrand.

$$x\sqrt{2x+3} \longrightarrow \frac{1}{2}(u-3)\sqrt{u}$$

- Step 3: Evaluate the transformed limits.

$$u(-1) = 2(-1) + 3 = 1 \quad \text{and} \quad u(1) = 2(1) + 3 = 5$$

- Step 4: Transform the integral and solve.

$$\begin{aligned}\int_{-1}^{1} x\sqrt{2x+3}\,\mathrm{d}x &= \frac{1}{2}\int_{-1}^{1} \overbrace{\frac{1}{2}(u-3)\sqrt{u}}^{f(u)}\;\overbrace{(2)}^{u'}\;\mathrm{d}x \\ &= \frac{1}{4}\int_{1}^{5}(u-3)u^{\frac{1}{2}}\,\mathrm{d}u \\ &= \frac{1}{4}\int_{1}^{5}\left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)\,\mathrm{d}u \\ &= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}}\right]_{1}^{5} \\ &= \frac{2}{5}\end{aligned}$$

6.2 Integration By Parts

This method is best applied to integrals of products of functions. The theorem is derived from the *Product Rule*,

$$(uv)' = uv' + vu' \quad \Longrightarrow \quad [uv]_a^b = \int_a^b uv'\,\mathrm{d}x + \int_a^b vu'\,\mathrm{d}x$$

Rearranging, we arrive at the formula for Integration by Parts.

$$\boxed{\int_a^b uv'\,\mathrm{d}x = [uv]_a^b - \int_a^b vu'\,\mathrm{d}x}$$

To guide our choice for u and v', we use the *DETAIL Rule*:

- **D**ashed should be tried in order of **E**xponentials, **T**rig functions, **A**lgebraic functions, **I**nverse trig functions, then **L**ogarithms.

6.2 Integration by Parts

For example, by integration by parts to evaluate $\int_0^\pi (1-x)\cos 3x\,\mathrm{d}x$.

- Step 1: Identify v' and integrate to find v.

$$v' = \cos 3x \quad \implies \quad v = \frac{1}{3}\sin 3x$$

- Step 2: Identify u and differentiate to find u'.

$$u = 1 - x \quad \implies \quad u' = -1$$

- Step 3: Use the formula for integration by parts to rewrite the integral.

$$\int_0^\pi (1-x)\cos 3x\,\mathrm{d}x = \left[(1-x)\left(\frac{1}{3}\sin 3x\right)\right]_0^\pi - \int_0^\pi \frac{1}{3}\sin 3x(-1)\,\mathrm{d}x$$

$$= 0 + \frac{1}{3}\int_0^\pi \sin 3x\,\mathrm{d}x$$

$$= \frac{1}{3}\left[-\frac{1}{3}\cos 3x\right]_0^\pi = \frac{2}{9}$$

6.3 Recurrence Relationships

Sometimes several applications of the Integration by Parts formula is needed to evaluate an integral. This often yields recurring expressions, which can be used to quickly get to the solution.

For example, evaluate $\int \mathrm{e}^{2x} \sin x \, \mathrm{d}x$.

$$\begin{aligned} v' &= \sin x &\implies&& v &= -\cos x \\ u &= \mathrm{e}^{2x} &\implies&& u' &= 2\mathrm{e}^{2x} \end{aligned}$$

Using one application of integration by parts,

$$\begin{aligned} \int \mathrm{e}^{2x} \sin x \, \mathrm{d}x &= \mathrm{e}^{2x} (-\cos x) - \int (-\cos x) \left(2\mathrm{e}^{2x}\right) \, \mathrm{d}x \\ &= -\mathrm{e}^{2x} \cos x + 2 \int \mathrm{e}^{2x} \cos x \, \mathrm{d}x \end{aligned}$$

6.3 Recurrence Relationships

Applying integration by parts a second time,

$$v' = \cos x \implies v = \sin x$$

$$\therefore \int \mathrm{e}^{2x} \sin x \, \mathrm{d}x = -\mathrm{e}^{2x} \cos x + 2 \left(\mathrm{e}^{2x} \sin x - \int \sin x \left(2\mathrm{e}^{2x} \right) \, \mathrm{d}x \right)$$

$$= -\mathrm{e}^{2x} \cos x + 2\mathrm{e}^{2x} \sin x - 4 \int \mathrm{e}^{2x} \sin x \, \mathrm{d}x$$

Noticing that $\int \mathrm{e}^{2x} \sin x \, \mathrm{d}x$ is a recurrence of the original integral, we can move it to the left-hand side.

$$5 \int \mathrm{e}^{2x} \sin x \, \mathrm{d}x = -\mathrm{e}^{2x} \cos x + 2\mathrm{e}^{2x} \sin x$$

$$\therefore \int \mathrm{e}^{2x} \sin x \, \mathrm{d}x = \frac{\mathrm{e}^{2x}}{5} \left(2 \sin x - \cos x \right) + C$$

6.3 Recurrence Relationships

Another powerful example occurs when u takes the form of an algebraic (or power) function, e.g. x^n.

Consider applying integration by parts to $I_n = \int x^n e^x \, dx$.

$$v' = e^x \implies v = e^x \quad \text{and} \quad u = x^n \implies u' = nx^{n-1}$$

$$\therefore \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

Written with different notation, we have

$$\boxed{I_n = x^n e^x - nI_{n-1}}$$

For example, $\int x^4 e^x \, dx$ can be evaluated using only algebra, since

$$\begin{aligned} I_4 &= x^4 e^x - 4I_3 \\ &= x^4 e^x - 4\left(x^3 e^x - 3\left(x^2 e^x - 2\left(xe^x - e^x\right)\right)\right) + C \\ &= e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C \end{aligned}$$

6.4 Rational Functions

When our integral involves the quotient of two functions, we can use the following approaches:

- Technique 1: Polynomial division, then integration term-by-term.
- Technique 2: Algebraic rearrangement to the form $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$, then

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$$

- Technique 3: Algebraic rearrangement to the form $\int \frac{1}{x^2+a^2}\,\mathrm{d}x$, then

$$\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

6.4 Rational Functions

For example, use Techniques 2 and 3 to evaluate $\int \frac{2x-3}{x^2-4x+5}\,\mathrm{d}x$.

$$\begin{aligned}
\int \frac{2x-3}{x^2-4x+5}\,\mathrm{d}x &= \int \frac{\overbrace{2x-4}^{f'(x)}+1}{x^2-4x+5}\,\mathrm{d}x \\
&= \int \frac{2x-4}{x^2-4x+5}\,\mathrm{d}x + \int \frac{1}{x^2-4x+5}\,\mathrm{d}x \\
&= \log|x^2-4x+5| + \int \frac{1}{x^2-4x+4+1}\,\mathrm{d}x \\
&= \log|x^2-4x+5| + \int \frac{1}{(x-2)^2+1}\,\mathrm{d}x \\
&= \log|x^2-4x+5| + \tan^{-1}(x-1) + C
\end{aligned}$$

6.5 Partial Fractions

Partial fractions is an extension of the previous techniques, applied to integrals of the form $\int \frac{P(x)}{Q(x)R(x)}\,\mathrm{d}x$.

The fraction is expanded into simpler (partial) fractions, which are then integrated term-by-term.

$$\boxed{\int \frac{P(x)}{Q(x)R(x)}\,\mathrm{d}x = \int \frac{A(x)}{Q(x)}\,\mathrm{d}x + \int \frac{B(x)}{R(x)}\,\mathrm{d}x}$$

For example, use the method of partial fractions to evaluate $\int \frac{2x-1}{x^2-5x+6}\,\mathrm{d}x$.

- Step 1: Factorise the denominator.

$$\frac{2x-1}{x^2-5x+6} = \frac{2x-1}{(x-2)(x-3)}$$

- Step 2: Split into partial fractions.

$$\frac{2x-1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

$$\implies 2x-1 = A(x-3) + B(x-2)$$

- Step 3: Substitute well-chosen values of x to find A and B.

$$x=2: \quad 2(2)-1 = A(2-3)+0 \qquad \therefore\ A=-3$$

$$x=3: \quad 2(3)-1 = 0 + B(3-2) \qquad \therefore\ B=5$$

- Step 4: Integrate term-by-term.

$$\int \frac{2x-1}{(x-2)(x-3)}\,\mathrm{d}x = \int \left(-\frac{3}{x-2} + \frac{5}{x-3}\right)\mathrm{d}x$$

$$= -3\ln|x-2| + 5\ln|x-3| + C$$

6.6 t Formulae Substitution

A special category of Integration by Substitution is to use $t = \tan\left(\frac{x}{2}\right)$ to solve integrals of the form,

$$\boxed{\int \frac{1}{a\cos x + b\sin x + c}\,\mathrm{d}x}$$

Defining $t = \tan\left(\frac{x}{2}\right)$, our essential substitution formulae are:

$$\mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2} \qquad \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2}$$

6.6 t Formulae Substitution

The first formula is derived using calculus as follows.

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\left(\frac{x}{2}\right)\right)\\ &= \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\\ &= \frac{1}{2}\left(\tan^2\left(\frac{x}{2}\right)+1\right)\\ &= \frac{1}{2}\left(t^2+1\right)\end{aligned}$$

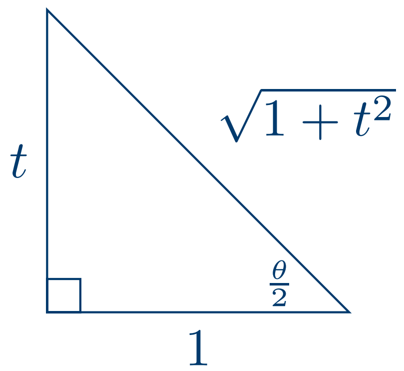
Separating the variables, we arrive at

$$\mathrm{d}t = \frac{1}{2}\left(t^2+1\right)\,\mathrm{d}x \quad \text{or} \quad \boxed{\mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}}$$

6.6 t Formulae Substitution

The remaining formulae are derived using geometry & double-angle identities.

Consider a right-angled triangle consistent with $t = \tan\frac{\theta}{2}$. With respect to the angle $\frac{\theta}{2}$, the ratio of opposite & adjacent sides is $t : 1$. Hence, by Pythagorus' Theorem, the hypotenuse is $\sqrt{1+t^2}$.



Using double-angle identities, we can express $\sin\theta$ in terms of t,

$$\begin{aligned}\sin\theta &= 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\\ &= 2\cdot\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}\\ &= \frac{2t}{1+t^2}\end{aligned}$$

Likewise for $\cos\theta$ and $\tan\theta$,

$$\begin{aligned}\cos\theta &= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\\ &= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2}\\ &= \frac{2t}{1-t^2}\end{aligned}$$

6.6 t Formulae Substitution

The t-formulae are used by substituting all expressions in x, *e.g.* $\mathrm{d}x$, $\sin x$, $\cos x$, and $\tan x$, for corresponding expressions in t.

For example,

$$\begin{aligned}
\int \frac{1}{3+2\cos x}\,\mathrm{d}x &= \int \frac{1}{3+2\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,\mathrm{d}t}{t^2+1} \\
&= 2\int \frac{1}{3(1+t^2)+2(1-t^2)}\,\mathrm{d}t \\
&= 2\int \frac{1}{5+t^2}\,\mathrm{d}t \\
&= 2\left(\frac{1}{\sqrt{5}}\right)\tan^{-1}\left(\frac{t}{\sqrt{5}}\right)+C \\
&= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan\left(\frac{x}{2}\right)\right)+C
\end{aligned}$$

Question 1 (3 Marks)

Evaluate

$$\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{2}{x \ln x}\, \mathrm{d}x.$$

Solution

Rewrite the integral in the form of $\int \frac{f'(x)}{f(x)}\, \mathrm{d}x$.

$$\begin{aligned}
\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{2}{x \ln x}\, \mathrm{d}x &= 2\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{\frac{1}{x}}{\ln x}\, \mathrm{d}x \\
&= 2\left[\ln|\ln x|\right]_{\mathrm{e}}^{\mathrm{e}^2} \\
&= 2\left(\ln\left|\ln \mathrm{e}^2\right| - \ln|\ln \mathrm{e}|\right) \\
&= 2(\ln 2 - \ln 1) \\
&= 2\ln 2
\end{aligned}$$

Question 2 (4 Marks)

Solve

$$\int_2^4 \frac{x^2}{12+9x^2}\,\mathrm{d}x.$$

Solution

Rewrite the integral in the form of $\int \frac{1}{x^2+a^2}\,\mathrm{d}x$.

$$\begin{aligned}\int_2^4 \frac{x^2}{12+9x^2}\,\mathrm{d}x &= \frac{1}{9}\int_2^4 \frac{9x^2+12-12}{12+9x^2}\,\mathrm{d}x \\ &= \frac{1}{9}\int_2^4 \left(1-\frac{4}{4+3x^2}\right)\mathrm{d}x \\ &= \frac{1}{9}\left[x-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right)\right]_2^4 \\ &= \frac{2}{81}\left(1-3\sqrt{3}\tan^{-1}\left(2\sqrt{3}\right)+\sqrt{3}\pi\right)\end{aligned}$$

Question 3 (4 Marks)

Evaluate

$$\int_1^{\frac{\sqrt{3}}{3}} \frac{1}{x^2\sqrt{x^2+1}}\,\mathrm{d}x.$$

Solution

Use integration by substitution, with $x = \tan u$.

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \sec^2 u \quad \therefore \quad \mathrm{d}x = \sec^2 u\,\mathrm{d}u$$

Substitute every x with u in the integrand.

$$\frac{1}{x^2\sqrt{x^2+1}} \longrightarrow \frac{1}{\tan^2 u\sqrt{\tan^2 u+1}} = \frac{1}{\tan^2 u \sec u}$$

6.7 HSC-Adapted Questions

When $x = 1$, $u = \tan^{-1}(1) = \frac{\pi}{4}$. When $x = \frac{\sqrt{3}}{3}$, $u = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$.

Transforming the integral, we have

$$\begin{aligned}\int_1^{\frac{\sqrt{3}}{3}} \frac{1}{x^2\sqrt{x^2+1}}\,\mathrm{d}x &= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{1}{\tan^2 u \sec u} \cdot \sec^2 u\,\mathrm{d}u \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{\cos u}{\sin^2 u}\,\mathrm{d}u \\ &= \Big[(-\sin u)^{-1}\Big]_{\frac{\pi}{4}}^{\frac{\pi}{6}} \\ &= -\frac{1}{\sin\left(\frac{\pi}{6}\right)} + \frac{1}{\sin\left(\frac{\pi}{4}\right)} \\ &= \sqrt{2} - 2\end{aligned}$$

Question 4 (5 Marks)

Using the substitution $t = \tan \frac{x}{2}$, or otherwise, show that

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{\sin x + \tan x} \, \mathrm{d}x = \log_e \sqrt{\frac{\sqrt{3}}{e}}.$$

Solution

We need the following three t formulae:

$$\mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \text{and} \quad \tan x = \frac{2t}{1-t^2}.$$

Also, when $x = \frac{\pi}{2}$, $t = \tan \frac{\pi}{4} = 1$. When $x = \frac{2\pi}{3}$, $t = \tan \frac{\pi}{3} = \sqrt{3}$.

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{\sin x + \tan x}\,\mathrm{d}x = \int_1^{\sqrt{3}} \frac{1}{\frac{2t}{1+t^2} + \frac{2t}{1-t^2}} \cdot \frac{2\,\mathrm{d}t}{1+t^2}$$

$$= \int_1^{\sqrt{3}} \frac{(1+t^2)(1-t^2)}{2t - 2t^3 + +2t + 2t^3} \cdot \frac{2\,\mathrm{d}t}{1+t^2}$$

$$= 2\int_1^{\sqrt{3}} \left(\frac{1}{4t} - \frac{t^2}{4t}\right)\mathrm{d}t$$

$$= \frac{1}{2}\left[\ln|t| - \frac{1}{2}t^2\right]_1^{\sqrt{3}}$$

$$= \frac{1}{2}\left(\ln\sqrt{3} - 1\right)$$

$$= \frac{1}{2}\left(\ln\sqrt{3} - \ln \mathrm{e}\right)$$

$$= \ln\left(\frac{\sqrt{3}}{\mathrm{e}}\right)^{\frac{1}{2}}, \text{ as required.}$$

Question 5 (7 Marks)

Let

$$I_k = \int_0^{\pi} \cos^{2k+1} x \, dx,$$

where k is an integer, $k \geq 0$.

(i) Show that

$$I_k = -\pi + (2k+1) \int_0^{\pi} x \sin x \cos^{2k} x \, dx$$

for $k \geq 1$. (3 Marks)

(ii) Show also that

$$I_k = \frac{2k}{2k-1} I_{k-1}$$

for $k \geq 1$. (3 Marks)

(ii) Explain why $I_k = 0$ for $k \geq 1$. (1 Mark)

Solution

(i) Using integration by parts, we have

$$v' = 1 \implies v = x$$
$$u = \cos^{2k+1} x \implies u' = (2k+1)\cos^{2k} x(-\sin x)$$

Substituting these values into the integration by parts formula,

$$\begin{aligned} I_k &= \left[x \cos^{2k+1} x\right]_0^{\pi} - \int_0^{\pi} x(2k+1)\cos^{2k} x(-\sin x)\,\mathrm{d}x \\ &= \pi \cdot (-1)^{2k+1} + (2k+1)\int_0^{\pi} x \sin x \cos^{2k} x\,\mathrm{d}x \\ &= -\pi + (2k+1)\int_0^{\pi} x \sin x \cos^{2k} x\,\mathrm{d}x, \end{aligned}$$

since $2k+1$ is odd for all integers $k \geq 0$.

(ii) Rewrite the integral as $\int_0^\pi \cos^{2k} x \cos x \, \mathrm{d}x$ and use integration by parts.

$$v' = \cos x \implies v = -\sin x$$
$$u = \cos^{2k} x \implies u' = 2k \cos^{2k-1} x(-\sin x)$$

$$\begin{aligned}
\implies I_k &= \int_0^\pi \cos^{2k} x \cos x \, \mathrm{d}x \\
&= \left[-\cos^{2k} x \sin x\right]_0^\pi - \int_0^\pi (-\sin x) 2k \cos^{2k-1} x(-\sin x) \, \mathrm{d}x \\
&= -2k \int_0^\pi \cos^{2k-1} x(1 - \cos^2 x) \, \mathrm{d}x \\
&= -2k I_{k-1} + 2k I_k \\
\therefore \ (1-2k) I_k &= -2k I_{k-1} \\
I_k &= \frac{2k}{2k-1} I_{k-1}, \text{ as required.}
\end{aligned}$$

6.7 HSC-Adapted Questions

(iii) Using Part (ii), we have

$$I_k = \frac{2k}{2k-1} I_{k-1}$$

for $k \geq 1$.

Substituting $k = 1$ gives us,

$$\begin{aligned} I_1 &= \frac{2}{2-1} I_0 \\ &= \int_0^{\pi} \cos x \, \mathrm{d}x \\ &= 0. \end{aligned}$$

Since I_k is a multiple of I_{k-1} for all integers $k \geq 1$, I_k is a multiple of zero. Hence, $I_k = 0$ for all integers $k \geq 1$.